Hamiltonian systems, Titchmarsh-Weyl coefficients, and models

Dijksma, A.

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DIJKSMA, A.

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This note concerns an eigenvalue problem for a Hamiltonian system of ordinary differential equations in an L^2-space with a boundary condition depending linearly on the eigenvalue parameter. We show that the spectral properties (in particular, the embedded eigenvalues) of this problem can be obtained from the Titchmarsh–Weyl coefficients. These coefficients appear in formulas for the generalized resolvent associated with a selfadjoint linearization of the problem in a Pontryagin space. They are generalized Nevanlinna functions and have representations in terms of selfadjoint relations (models) and integral representations. The note is based on the joint paper [1].

1. The system of differential equations

We consider the 2×2 system of first order ordinary differential equations

$$Jf'(t) - H(t)f(t) = \lambda\Delta(t)f(t) + \Delta(t)g(t), \text{ almost all } t \in [0,\infty). \tag{1}$$

We assume (a)–(f): (a) The system is *Hamiltonian*, that is, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. (b) The 2×2 matrix functions H and Δ are locally integrable on $[0,\infty)$; in particular, (1) is *regular* at 0, which means that the two matrix functions are integrable over $[0,1]$, say. (c) The system is *formally symmetric*, that is, $H(t)^* = H(t)$ and $\Delta(t)^* = \Delta(t)$, $t \in [0,\infty)$. (d) $\Delta \geq 0$ almost everywhere on $[0,\infty)$; so, the Hilbert space $L^2(\Delta dt)$ of equivalence classes of 2-vector functions with inner product $(f,g) = \int_0^\infty g(t)^*\Delta(t)f(t)dt$ can be defined as usual. (e) The system is *definite*, which means that if $Jf' - Hf = 0$ and $\Delta f = 0$ on $[0,\infty)$, then $f = 0$ on $[0,\infty)$. (f) The system is in the limit point case, which we explain in a moment.

We use the following notions. A *(closed) linear relation* T in a Hilbert or Pontryagin space $\mathcal{H}$ is a (closed) linear subset of $\mathcal{H} \oplus \mathcal{H}$. The *multivalued part* of T is denoted by $T(0) = \{g : \{0,g\} \in T\}$ and its *adjoint* is the closed linear relation $T^* = \{\{h,k\} : (k,f) - (h,g) = 0, \forall\{f,g\} \in T\}$.

The set of all pairs $\{f,g\}$ of equivalence classes $f,g \in L^2(\Delta dt)$ such that f contains a locally absolutely continuous function $\tilde{f}$ and g contains a function $\tilde{g}$ such that $J\tilde{f}' - H\tilde{f} = \Delta\tilde{g}$ almost everywhere on $[0,\infty)$ is called the *maximal relation* associated with (1). It is a closed linear relation in $L^2(\Delta dt)$ and is denoted by T_{max}. The linear relation $T_{min} = T_{max}^*$ is called the *minimal relation* associated with (1); it is symmetric, that is, contained in its adjoint. The *limit point condition* is equivalent to the condition that for all $\lambda \in \mathbf{C} \setminus \mathbb{R}$, $\ker(T_{max} - \lambda I) = \{f : \{f, \lambda f\} \in T_{max}\}$ is 1-dimensional. Let $Y(t,\lambda) = (y_{ij}(t,\lambda))_{i,j=1,2}$, $Y(0,\lambda) = I$, be the fundamental solution of $Jf' - Hf = \lambda\Delta f$. Then the limit point condition implies that the *Titchmarsh–Weyl coefficient* m:

$$m(\lambda) = \lim_{t\to\infty} - y_{11}(t,\lambda)/y_{12}(t,\lambda)$$

is a well defined function on $\mathbf{C} \setminus \mathbb{R}$ and that $\ker(T_{max} - \lambda I)$ is spanned by the 2-vector function

$$\mathcal{Y}(t,\lambda) = \begin{pmatrix} y_{11}(t,\lambda) + m(\lambda)y_{12}(t,\lambda) \\ y_{21}(t,\lambda) + m(\lambda)y_{22}(t,\lambda) \end{pmatrix}.$$

Moreover, $T_{min} = \{\{f,g\} \in T_{max} : \tilde{f}_1(0) = 0, \tilde{f}_2(0) = 0\}$. From now on we write f, f_1 and f_2 instead of $\tilde{f}$ (the locally absolutely continuous representative of f), and its components $\tilde{f}_1$ and $\tilde{f}_2$.

2. The boundary condition

Let $\lambda \in \mathbf{C} \setminus \mathbb{R}$ and $g \in L^2(\Delta dt)$. We look for solutions $f \in \operatorname{dom} T_{max}$ of (1) that satisfy the *boundary condition*

$$\alpha(\lambda)f_1(0) + \beta(\lambda)f_2(0) = 0, \quad \alpha(\lambda) = \alpha_0 - \alpha_1\lambda, \quad \beta(\lambda) = \beta_0 - \beta_1\lambda. \tag{2}$$

Here α_0, α_1, β_0, and β_1 are real numbers satisfying $d = \alpha_0\beta_1 - \alpha_1\beta_0 \neq 0$, $\alpha(\mu) + \beta(\mu)m(\mu) \neq 0$ for some $\mu \in \mathbf{C}^+$. The joint problem (1), (2) is a *boundary eigenvalue problem (BEP)* with boundary condition containing the *eigenvalue* *parameter* λ. The boundary condition does not depend on λ if $d = 0$. Related to $m(\lambda)$ is the case where $\alpha_0 = 1$ and $\alpha_1 = \beta_0 = \beta_1 = 0$; see Theorem 1(i) below. Then (2) becomes simply $f_1(0) = 0$, and the relation

$$A_0 = \{\{f, g\} \in T_{max} : f_1(0) = 0\}$$

is selfadjoint in the space $L^2(\Delta dt)$, that is, equal to its adjoint.

3. A linearization

The BEP (1), (2) admits the selfadjoint linearization

$$A_{lin} = \{\{(f,\ \alpha_1 f_1(0) + \beta_1 f_2(0)), (g,\ \alpha_0 f_1(0) + \beta_0 f_2(0))\} : \{f, g\} \in T_{max}\}.$$

It is a selfadjoint relation in the space $L^2(\Delta dt) \oplus \mathbf{C}$ equipped with the inner product

$$((f,\ \varphi), (g,\ \psi)) = \int_0^\infty g(t)^* \Delta(t) f(t) dt + \overline{\psi} \frac{1}{d} \varphi, \quad f, g \in L^2(\Delta dt), \quad \varphi, \psi \in \mathbf{C}.$$

If $d > 0$, this space is a Hilbert space; if $d < 0$, it is a Pontryagin space with negative index 1.
Denote by $P_{L^2(\Delta dt)}$ the orthogonal projection from $L^2(\Delta dt) \oplus \mathbf{C}$ onto $L^2(\Delta dt)$. Then the operator

$$R(\lambda) = P_{L^2(\Delta dt)}(A_{lin} - \lambda I)^{-1} \mid_{L^2(\Delta dt)}, \quad \lambda \in \rho(A_{lin}), \tag{3}$$

is bounded on $L^2(\Delta dt)$ and locally holomorphic on the resolvent set $\rho(A_{lin})$. A_{lin} is called a *linearization* of the BEP (1), (2) because for every $g \in L^2(\Delta dt)$, the function $f = R(\lambda)g$ is the (unique) solution of (1), (2). The operator (3) is called the *generalized resolvent* of A_{lin}. The resolvent of A_{lin} is given by

$$(A_{lin} - \lambda I)^{-1}(g, \psi) = (f,\ \alpha_1 f_1(0) + \beta_1 f_2(0)), \quad f = R(\lambda)g + \frac{\mathcal{Y}(\cdot, \lambda)\psi}{\alpha(\lambda) + \beta(\lambda) m(\lambda)}. \tag{4}$$

A nonreal point λ_0 is an eigenvalue of A_{lin} if and only if $y(t, \lambda_0) = \mathcal{Y}(t, \lambda_0)$ satisfies (2) for $\lambda = \lambda_0$. A real number λ_0 is an eigenvalue of A_{lin} if and only if there is a neighborhood $\mathcal{U}_0$ of λ_0 and a holomorphic function $a(\lambda)$ on $\mathcal{U}_0 \cap \mathbf{C}^+$ such that the limits $a_1 = \lim_{\eta \downarrow 0} a(\lambda_0 + i\eta)$, $a_2 = \lim_{\eta \downarrow 0} m(\lambda_0 + i\eta) a(\lambda_0 + i\eta)$ exist and $y(t, \lambda_0) = Y_1(t, \lambda_0) a_1 + Y_2(t, \lambda_0) a_2$ satisfies (2) for $\lambda = \lambda_0$. The eigenvalue λ_0 of A_{lin} is of *nonpositive (postive, neutral) type* if the corresponding eigenfunction $(y(\cdot, \lambda_0),\ \alpha_1 y_1(0, \lambda_0) + \beta_1 y_2(0, \lambda_0))$ is nonpositive (postive, neutral, respectively).

4. The generalized resolvent

We give two integral representations for the function $R(\lambda)g$, where $g \in L^2(\Delta dt)$ has compact support. For the first one we write $Y = (Y_1 \quad Y_2)$, that is, the first (second) column of Y is denoted by Y_1 (Y_2). Then

$$R(\lambda)g(t) = Y(t, \lambda)\Omega(\lambda) \int_0^\infty Y(x, \overline{\lambda})^* \Delta(x) g(x) dx$$
$$-Y_2(t, \lambda) \int_0^t Y_1(x, \overline{\lambda})^* \Delta(x) g(x) dx - Y_1(t, \lambda) \int_t^\infty Y_2(x, \overline{\lambda})^* \Delta(x) g(x) dx,$$

where the terms which are not entire in λ are collected in the 2×2 matrix function

$$\Omega(\lambda) = -(\alpha(\lambda) + \beta(\lambda) m(\lambda))^{-1} \begin{pmatrix} \beta(\lambda) & -\alpha(\lambda) \\ -\alpha(\lambda) & -\alpha(\lambda) m(\lambda) \end{pmatrix}.$$

The second integral formula is obtained by regrouping these nonentire functions into one function:

$$\omega(\lambda) = d^{-1} \frac{\alpha_1 + \beta_1 m(\lambda)}{\alpha(\lambda) + \beta(\lambda) m(\lambda)}.$$

Indeed, if we set $(Z \quad W)(t, \lambda) = (Y_1 \quad Y_2)(t, \lambda) \begin{pmatrix} \beta_1 d^{-1} & -\beta(\lambda) \\ -\alpha_1 d^{-1} & -\alpha(\lambda) \end{pmatrix}$ then

$$R(\lambda)g(t) = W(t, \lambda)\omega(\lambda) \int_0^\infty W(x, \overline{\lambda})^* \Delta(x) g(x) dx$$
$$+Z(t, \lambda) \int_0^t W(x, \overline{\lambda})^* \Delta(x) g(x) dx + W(t, \lambda) \int_t^\infty Z(x, \overline{\lambda})^* \Delta(x) g(x) dx.$$

The functions Ω and ω are also called Titchmarsh–Weyl functions, just like the function m.

5. Nevanlinna functions and models

A $p \times p$ matrix function $Q(\lambda)$ belongs to the class N_κ^p (N_κ if $p = 1$) if it is meromorphic on $\mathbf{C} \setminus \mathbb{R}$, $Q(\overline{\lambda}) = Q(\lambda)^*$, and the kernel $N_Q(z,\lambda) = \frac{Q(z)-Q(\lambda)^*}{z-\overline{\lambda}}$ has κ negative squares. This means that all square matrices of the form $(f_j^* N_Q(\lambda_i,\lambda_j)f_i)_{i,j=1,\dots,k}$, where the λ_i's are points of holomorphy of Q in $\mathbf{C}^+$ and the f_i's are p-vectors, have at most κ negative eigenvalues and at least one such matrix has exactly κ negative eigenvalues, counted with their multiplicities. The functions in N_0^p are called *Nevanlinna functions*; they are locally holomorphic on $\mathbf{C} \setminus \mathbb{R}$. The functions in N_κ^p with $\kappa > 0$ are called *(generalized) Nevanlinna functions with κ negative squares.* Since $N_m(z,\lambda) = (\mathcal{Y}(\cdot,z), \mathcal{Y}(\cdot,\lambda))$, m is a Nevanlinna function. In the Hilbert space case ($d > 0$) Ω and ω are Nevanlinna functions also; in the Pontryagin space case ($d < 0$) $\Omega \in N_1^2$ and ω can belong to either N_0 or N_1.
Every $Q \in N_\kappa^p$ admits a representation of the form

$$Q(\lambda) = C + \Gamma^*[(\lambda - \operatorname{Re}\mu) + (\lambda - \overline{\mu})(\lambda - \mu)(A - \lambda I)^{-1}]\Gamma,$$

where C is a Hermitian $p \times p$ matrix, A is a selfadjoint relation in a Pontryagin space $\mathcal{P}$ with negative index κ with nonempty resolvent set $\rho(A)$, $\mu \in \rho(A)$, $\Gamma : \mathbf{C}^p \to \mathcal{P}$ is a mapping, and a *minimality condition* holds: $\{(A - \lambda I)^{-1}\Gamma x : \lambda \in \rho(A),\ x \in \mathbf{C}^p\} = \mathcal{P}$. This model is unique up to isomorphism. To denote the dependence on Q we write $A(Q)$, $\mathcal{P}(Q)$ etc.; the spectral family of $A(Q)$ will be written as E_Q.

Theorem 1.
(i) $(A(m), \mathcal{P}(m)) \cong (A_0, L^2(\Delta dt))$, *in particular, $\mathcal{P}(m)$ is a Hilbert space.*
(ii) $(A(\Omega), \mathcal{P}(\Omega)) \cong (A_{lin}|_{T_{min}(0)^\perp}, (L^2(\Delta dt) \ominus T_{min}(0)) \oplus \mathbf{C})$.
(iii) $(A(\Omega)|_{\operatorname{ran} E_\Omega(\iota)}, \operatorname{ran} E_\Omega(\iota)) \cong (A(\omega)|_{\operatorname{ran} E_\omega(\iota)}, \operatorname{ran} E_\omega(\iota))$ *for every compact interval $\iota \subset \mathbb{R}$ whose endpoints do not coincide with the critical points of $A(\Omega)$ and $A(\omega)$.*
(iv) *The root spaces corresponding to the nonreal eigenvalues of A_{lin}, $A(\Omega)$ and $A(\omega)$ coincide.*
In particular, A_{lin}, $A(\Omega)$ and $A(\omega)$ have the same finite (that is, excluding the point ∞) point and continuous spectrum. If $A(\Omega)$ and $A(\omega)$ are densely defined selfadjoint operators, then $(A(\Omega), \mathcal{P}(\Omega)) \cong (A(\omega), \mathcal{P}(\omega))$, and in particular, the negative index of $\mathcal{P}(\Omega)$ is equal to the negative index of $\mathcal{P}(\omega)$.

6. The spectrum of the boundary eigenvalue problem

In the sequel we only consider the scalar function ω; the results related to Ω are similar. Functions in N_κ have an integral representation, which can be obtained from their model. For example, Q belongs to N_0 if and only if there are real numbers a_0, a_1 with $a_1 \geq 0$, and a bounded nondecreasing rightcontinuous function σ such that

$$Q(\lambda) = a_0 + a_1\lambda + \int_{-\infty}^{\infty} \frac{1+t\lambda}{t-\lambda} d\sigma(t). \tag{5}$$

If $a_1 < 0$, then Q defined by this formula belongs to class N_1, but the integral representation of N_1-functions is more complicated than this. The general formula is given by M.G. Kreĭn and H. Langer.
From now on we set in (2): $\beta_0 = 1$, $\beta_1 = 0$; then $d = -\alpha_1$. Assume that m has representation (5). Then

$$\omega(\lambda) = -\left(b_0 + b_1\lambda + \int_{-\infty}^{\infty} \frac{1+t\lambda}{t-\lambda} d\sigma(t)\right)^{-1}, \quad b_0 = a_0 + \alpha_0,\ b_1 = a_1 - \alpha_1,$$

and the limit $r = \lim_{y\to\infty} \omega(iy)/iy$ exists. It can be shown that $A(\omega)$ is not an operator (that is, $A(\omega)(0)$ is a nontrivial subspace of $\mathcal{P}(\omega)$) if and only if $r \neq 0$, or equivalently, $s = \int_{-\infty}^{\infty}(1+t^2)d\sigma(t) < \infty$, $b_0 = \int_{-\infty}^{\infty} t\,d\sigma(t)$ and $b_1 = 0$ (and then $rs = 1$). We also have that

$$\lim_{\eta\to\infty} (-i\eta)\,\omega(i\eta) = b_1^{-1}, \quad b_1 \neq 0. \tag{6}$$

Consider the case $b_1 \geq 0$. Then $\omega \in N_0$ and $\mathcal{P}(\omega)$ is a Hilbert space. If $b_1 > 0$ then $A(\omega)$ is an operator and by (6), $\omega(\lambda) = \int_{-\infty}^{\infty} \frac{d\tau(t)}{t-\lambda}$ for some bounded nondecreasing function τ. If, in addition, $\alpha_1 > 0$ then $L^2(\Delta dt) \oplus \mathbf{C}$ is a Pontryagin space, and therefore $A(\omega)$ does not contain information about the spectral behavior of A_{lin} at ∞.
Now consider the case $b_1 < 0$. Then $d < 0$, $\omega \in N_1$, and (6) implies that ∞ cannot be a pole of ω of nonpositive type. By (6) and because N_1 functions either have a pair $\lambda_0, \overline{\lambda_0}$, say, of nonreal simple poles or have one pole of nonpositive type in a point $\lambda_0 \in \mathbb{R} \cup \{\infty\}$, ω has one of the following three forms:

$$\omega(\lambda) = \begin{cases} \frac{c}{\lambda_0-\lambda} + \frac{\overline{c}}{\overline{\lambda_0}-\lambda} + \int_{-\infty}^{\infty} \frac{d\tau(t)}{t-\lambda}, & \text{(case 1)} \\ \sum_{j=1}^{3} \frac{c_j}{(\lambda_0-\lambda)^j} + \int_{-\infty}^{\infty} \frac{d\tau(t)}{t-\lambda}, & \text{(case 2)} \\ \sum_{j=1}^{3} \frac{c_j}{(\lambda_0-\lambda)^j} + \int_{|t-\lambda_0|\leq\delta} \left(\frac{1}{t-\lambda} + \sum_{j=1}^{2} \frac{(t-\lambda_0)^{j-1}}{(\lambda-\lambda_0)^j}\right) \frac{d\tau(t)}{(t-\lambda_0)^2} + \int_{|t-\lambda_0|\geq\delta} \frac{d\tau(t)}{t-\lambda}, & \text{(case 3)} \end{cases}$$

where in case 1: $\lambda_0 \neq \overline{\lambda_0}$, $0 \neq c \in \mathbf{C}$ and τ is a bounded nondecreasing function; in case 2: $\lambda_0 = \overline{\lambda_0}$, $c_3 \geq 0$, $|c_2| + |c_3| \neq 0$, or, if $c_2 = c_3 = 0$, $c_1 < 0$, and τ is a bounded nondecreasing function continuous in $t = \lambda_0$; and in case 3: $\lambda_0 = \overline{\lambda_0}$, $c_3 \geq 0$, $\delta > 0$ is arbitrarily chosen, τ is a bounded nondecreasing function continuous in $t = \lambda_0$ and such that $\int_{|t-\lambda_0|\leq\delta}(t-\lambda_0)^{-2}d\tau(t) = \infty$. From (6) it follows for the cases 1, 2, and 3, respectively, that

$$2\operatorname{Re} c + \int_{-\infty}^{\infty} d\tau(t) = b_1^{-1}, \quad c_1 + \int_{-\infty}^{\infty} d\tau(t) = b_1^{-1}, \quad c_1 + \int_{|t-\lambda_0|\geq\delta} d\tau(t) = b_1^{-1}.$$

Besides the eigenvalue λ_0, and, possibly, $\overline{\lambda_0}$, the spectrum of $A(\omega)$ coincides with the support of τ (that is, the complement of the union of the open intervals on which τ is constant), and the eigenvalues of $A(\omega)$ are the points of discontinuity of τ. By Theorem 1, this then completely describes the finite spectrum of A_{lin} and hence of the BEP (1), (2). Eigenvalues of this BEP can be characterized in another way.

Theorem 2.

(i) $\lambda_0 \in \mathbf{C}^+ \cap \sigma_p(A(\omega))$ *if and only if it is a (and then the only) zero of* ω^{-1} *in* $\mathbf{C}^+$.

(ii) $\lambda_0 \in \mathbb{R} \cap \sigma_p(A(\omega))$ *if and only if* $\lim_{\eta\downarrow 0}\eta^{-1}\omega(\lambda_0 + i\eta)^{-1} = 0$, *or equivalently,*

$$\int_{-\infty}^{\infty} \frac{d\sigma(t)}{(t-\lambda_0)^2} < \infty, \quad b_0 + b_1\lambda_0 + \int_{-\infty}^{\infty} \frac{1+t\lambda_0}{t-\lambda_0} d\sigma(t) = 0.$$

The eigenvalue is of nonpositive (positive) type if and only if $b_1 + \int_{-\infty}^{\infty} \frac{(t^2+1)d\sigma(t)}{(t-\lambda_0)^2} \leq 0$ *(*> 0*, respectively).*

7. An example

Let σ be a bounded nondecreasing function such that in a real neighborhood $\mathcal{U}_0$ of $\lambda_0 \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} \frac{d\sigma(t)}{(t-\lambda_0)^2} < \infty, \quad \int_{-\infty}^{\infty} \frac{d\sigma(t)}{(t-\lambda)^2} = \infty \text{ for all } \lambda \in \mathcal{U}_0 \setminus \{\lambda_0\}. \tag{7}$$

By a theorem of L. de Branges there exists a system (1) satisfying (a)–(f) such that the Titchmarsh–Weyl function m is of the form (5) with $a_1 = 0$ and some real a_0. Choose in (2) $\beta(\lambda) = 1$, and $\alpha_1 \neq 0$, α_0 such that

$$(a_0 + \alpha_0) - \alpha_1\lambda_0 + \int_{-\infty}^{\infty} \frac{1+t\lambda_0}{t-\lambda_0} d\sigma(t) = 0, \quad -\alpha_1 + \int_{-\infty}^{\infty} \frac{(t^2+1)d\sigma(t)}{(t-\lambda_0)^2} \leq 0 \ (> 0).$$

Then λ_0 is an eigenvalue of nonpositive (positive, respectively) type and $\lambda_0 \in \mathcal{U}_0 \setminus \{\lambda_0\}$ is not an eigenvalue of the BEP (1), (2).

The eigenvalues of nonpositive type behave differently under small perturbations of the parameters: Replace in the boundary condition α_0 by $\alpha_0 + \varepsilon$ and denote the corresponding linearization by A_{lin}^{ε}. The space $L^2(\Delta dt) \oplus \mathbf{C}$ does not depend on α_0 and ε, and by (4), the resolvent of A_{lin}^{ε} depends continuously in the uniform topology on ε in a neighborhood of 0 and $A_{lin}^{0} = A_{lin}$. A theorem of H. Langer and B. Najman implies that the eigenvalues $\lambda(\varepsilon)$ of A_{lin}^{ε} of nonpositive type depend continuously on ε. By Theorem 2 (ii) and (7) they cannot be real, so they must appear in conjugate pairs and are of neutral type. If the eigenvector $y(0)$ of A_{lin} at λ_0 is negative, then the eigenvectors $y(\varepsilon)$ of A_{lin}^{ε} at $\lambda(\varepsilon)$ cannot be continuous at $\varepsilon = 0$ as they are neutral.

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8. References

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Address: A. DIJKSMA, UNIVERSITY OF GRONINGEN, DEPARTMENT OF MATHEMATICS, P.O. BOX 800, 9700 AV GRONINGEN, THE NETHERLANDS.